Electricity and magnetism 2

Instructor: A.M. van den Berg

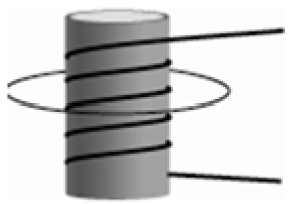
You don't have to use separate sheets for every question.
Write your name and S number on every sheet
There are **6 questions** with a total number of marks: 100

WRITE CLEARLY

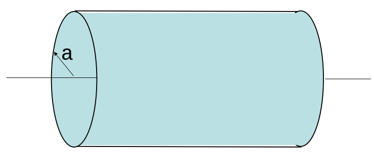
(1) (Total 8 marks)
A solenoid is made from a long wire, twisted in 5 turns around a tube. A current I_s flows through this wire. The length of the solenoid is 38 cm and the diameter of the solenoid is 20 cm.

(a) (4 marks)
Calculate on the axis of the solenoid the magnetic field under the assumption that the solenoid has infinite length and that the value of the current is given as $I_s = 80$ mA.

(b) (4 marks)
One puts around the solenoid a current loop. The plane of the current loop is perpendicular to the axis of the solenoid. The current is a single loop with a diameter of 25 cm and it has a resistance $R = 12\ \Omega$. One changes the current I_s from 80 mA to 300 mA. The time needed to make this change is exactly 2 s. Calculate the induced current in the current loop.



(2) (Total 20 marks)
A coaxial cable consists of a core and a mantle. The radius of the mantle has a value a. A current I flows through the core of the conductor and back through the mantle. The current changes in time as: $I = I_0 \cos(\omega t)$. For all practical purposes, you may assume that the wire is infinitely thin and that also the mantle is infinitely thin, i.e., the currents run only at distances $r = 0$ (wire) and $r = a$ (mantle).



(a) (10 marks) Give an expression of the magnetic field strength for $r > a$ and for $0 < r < a$.
(b) (10 marks) Calculate the induced electric field strength for $r > a$ and for $0 < r < a$.

(3) (Total 10 marks)
A parallel-plate capacitor, at rest in S_0 and tilted at a 45° angle to the x_0 axis, carries charge densities $\pm\sigma_0$ on the two plates. System S is moving to the right at speed v relative to S_0.

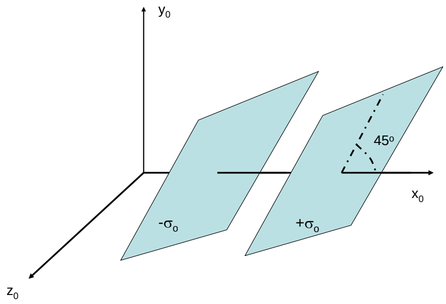
(a) (5 marks)
Find $\vec{E}_0$, the field in S_0

(b) (5 marks)
Find $\vec{E}$, the field in S

Hints: The Lorentz transformations for the electric field are given as:

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}$$
$$\vec{E}'_{\perp} = \gamma\left(\vec{E} + \vec{v} \times \vec{B}\right)_{\perp}$$

Please note, please provide $\vec{E}$ and $\vec{E}_0$ in the three coordinates x, y, and z.

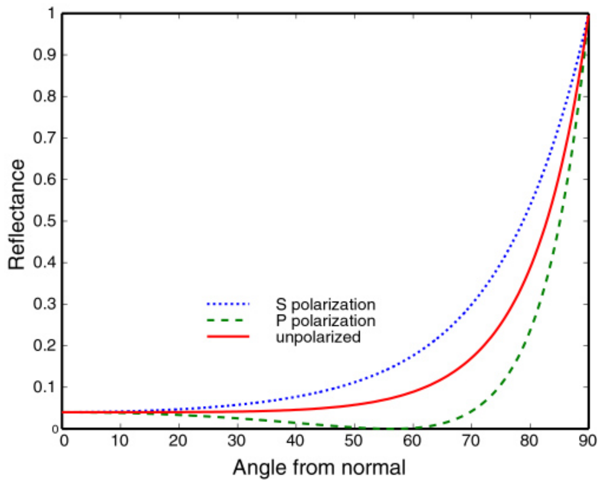


(4) (Total 20 marks)
Light from the sun can be assumed to be unpolarized. Therefore, the intensity of incoming light with the polarization vector perpendicular to the plane of incidence is equal to the intensity of the incoming light with the polarization direction parallel to the plane of incidence. In the figure the reflection coefficient is displayed for the scattering of electromagnetic waves on the air-water interface for these two polarization directions. In an experiment, unpolarized light from the sun is reflected on the surface of a lake.

(a) (4 marks)
A researcher wants to use this reflected sunlight to get a beam of fully polarized light. Use the figure to estimate the angle between the sun and the horizon to optimize the conditions for such an experiment.

(b) (4 marks)
What will be the direction of the polarization in that case?

(c) (4 marks)
What will happen with the degree of polarization and the direction of the polarization for the reflected light if the surface of the lake is covered by a highly-polished plate of Aluminum ?



Another researcher cannot perform his experiment which requires polarized light close to such a lake. Furthermore, he doesn't have polarization filters. Therefore, he decides to use the sunlight reflected by the molecules in the atmosphere of the Earth to get polarized light.

(d) (4 marks)
Make a drawing of a possible setup for such an experiment.

(e) (4 marks)
Explain why your setup can be used to perform this experiment.

(5) (Total 10 marks)
The fourth Maxwell equation is:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \vec{J}_d = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

To directly measure the displacement current density J_d, researchers use a time-varying voltage to charge and discharge a circular parallel-plate capacitor.

(a) (5 marks)
Find the displacement current density as a function of time that would produce a magnetic field given by:

$$\vec{B} = \frac{r\,\omega\,\Delta V\ \cos(\omega\,t)}{2\,d\,c^2}\hat{\phi}$$

where r is the distance from the center of the capacitor, ω is the angular frequency of the applied voltage ΔV, d is the plate spacing, and c is the speed of light.

(b) (5 marks)
Find an expression for the electric field $\vec{E}$.

(6) (32 marks)
Consider a capacitor, which is constructed by two parallel circular plates. Each plate is connected at the center of the plates to a thin wire. The capacitor is charged through the wires by a constant current I. The radius of the plates is given as a and the distance between is given as w, with $w \ll a$, i.e. you can ignore edge effects. Assume that the current flows out over the plates in such a way that the charge density σ is uniform at any given time, and is zero at $t = 0$.

(a) (4 marks)
Find the electric field between the plates as function of time t and current I.

(b) (4 marks)
Find the displacement current through a circle of radius s in the plane midway between the two plates.

(c) (4 marks)
Using this circle with radius s as your Amperian loop, and the flat surface that spans it, find the magnetic field between the plates at a distance s from the axis.

(d) (4 marks)
Find the electromagnetic energy density u_{EM} in the gap

(e) (4 marks)
Find the Poynting vector S in the gap. And proof that

$$\frac{\partial u_{EM}}{\partial t} = -\vec{\nabla} \circ \vec{S}$$

(f) (4 marks)
Determine the total electromagnetic energy contained in the gap as a function of time.

(g) (4 marks)
Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface.

(h) (4 marks)
Check that the power is equal to the rate of increase of the energy in the gap.

Electricity and magnetism 2: SOLUTIONS

(1) Question 1

(a) The magnetic field of an infinity long solenoid is homogeneous inside the solenoid and equal to 0 outside the solenoid. The field strength inside (on thus on the axis) is given as:

$$\vec{B} = \frac{\mu_0\, N\, I_s}{\ell}\hat{z}$$

where N is the number of turns (5) and ℓ is the length of the solenoid (0.38 m). For a current $I_s = 80$ mA, we can calculate:

$$B = \frac{4\pi \times 10^{-7} \times 5 \times 80 \times 10^{-3}}{0.38} = 1.32 \times 10^{-6}\ \text{T}$$

(b) The **emf** is given by Faraday's law:

$$\text{emf} = -\frac{d}{dt}\int_S \vec{B}\cdot\hat{n}\, da$$

This is the equation for the change in the magnetic flux. Keep in mind that the flux is contained to the diameter of the solenoid, so the integration over the surface is limited to $d = 20$ cm or $r_s = 10$ cm;

$$\text{emf} = -\pi r_s^2 \frac{dB}{dt} = -\pi r_s^2 \frac{\mu_0\, N}{\ell}\frac{dI_s}{dt}$$

The question was lacking the value of dt (don't worry I was flexible in my grading here), but if we assume that $dt = 2$ s and given that $dI = 80 - 300$ mA = -0.220 A, we can calculate the emf as:

$$\text{emf} = \pi \times 0.1^2\ \frac{4\,\pi \times 10^{-7} \times 5}{0.38}\frac{(-0.22)}{2} = -5.7 \times 10^{-8}\text{V}$$

With the given value of $R = 12\ \Omega$, we find that $I_{loop} = \text{emf}/R = 4.8 \times 10^{-9}$ A

(2) Question 2

(a) Here I will assume that the forth and back running currents are in phase and that the cross sections of the core and of the mantle are infinitely small; in other words the currents are limited to the center $s = 0$ (forth) and $s = a$ (back). Again, I'll be flexible in grading. Use the law of Ampère in both ranges $s < a$ and $s > a$.
For $0 < s < a$, we find

$$\oint \vec{B} \circ d\vec{\ell} = \mu_0 I_{enclosed}$$

Thus in this case:

$$B = \frac{\mu_0}{2\pi\ s} I_0 \cos(\omega t)$$

Because of symmetry reasons, the magnetic field field vector is in the ϕ direction.

$$\vec{B} = \frac{\mu_0}{2\pi\ s} I_0 \cos(\omega t)\hat{\phi}$$

For $s > a$, the enclosed current is zero, thus here we find $B = 0$.

(b) The magnetic field changes only in the region $0 < s < a$, for $s > a$ there will be no induced electric field. To calculate the induced electric field for $0 < s < a$, we use the equation: $\vec{\nabla} \times \vec{E} = -\partial\vec{B}/\partial t$, where we note that only the ϕ direction of the B-field gives a contribution. Furthermore, because of symmetry, the E-field will be in the z-direction. Using the curl operator in cylindrical coordinates we find:

$$\left(\vec{\nabla} \times \vec{E}\right)_\phi = \left[\frac{\partial E_s}{\partial z} - \frac{\partial E_z}{\partial s}\right]_\phi$$

Thus we can write this as:

$$\frac{\partial E_z}{\partial s} = -\frac{\partial B_\phi}{\partial t} = -\frac{\mu_0 I_0}{2\pi s}\frac{\partial}{\partial t}\cos(\omega t) = \frac{\omega\mu_0 I_0}{2\pi s}\sin(\omega t)$$

We now integrate thus over s, where the only relevant part in the integral is $\frac{1}{s}$; the integral of this function is $\ln(s)$. Therefore, E_z is given as:

$$E_z = \frac{\omega\mu_0 I_0}{2\pi}\sin(\omega t)\left[A - \ln(s)\right]$$

Here A denotes an integration constant, which we can calculate easily, because we note that for $s = a$ the electric field must go to zero (remind the solution for $s > a$, we found that $E = 0$). Thus the value of $A = \ln(a)$, and

$$E_z = \frac{\omega\mu_0 I_0}{2\pi}\sin(\omega t)\left[\ln(a) - \ln(s)\right]$$

Alternatively, you can also calculate the line integral over the E-field in a rectangle with length L. Here the rectangle has one leg with length L parallel to the axis inside the radius a, the "return leg" is at radius $s > a$ and the two joints are in between. Note that these two joints and the path integral outside radius $s > a$ don't contribute, because for the "joints" the field is perpendicular to the path, and for the "return leg", the field is zero.

$$\oint \vec{E}\circ d\vec{\ell} = E\ L = -\frac{d}{dt}\Phi_B = -\frac{d}{dt}\int \vec{B}\circ\hat{n}\ da$$

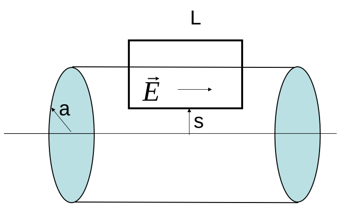
where da is the area of the loop (it is not a circle, because the loop runs parallel with the z-axis). Therefore,

$$E\ L = -\frac{d}{dt}\int_s^a\int_0^L \frac{\mu_0 I_0}{2\pi s'}\cos(\omega t) d\ell\ ds'$$

The integration over $d\ell$ and s' and the derivative with respect to t can be interchanged and the results is straight forward:

$$E = \frac{\mu_0 I_0}{2\pi}\omega\ln\frac{a}{r}\sin(\omega t)$$

And we remember that the E-field is in the z-direction only.



(3) Question 3

- The magnitude of the electric field in the system S_0 (this is the system where the capacitor is at rest) is
$|\vec{E}_0| = \dfrac{\sigma_0}{\epsilon_0}$.

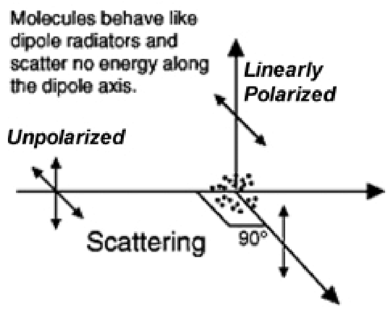
It points from the positive charges towards the negative charges and it is perpendicular to the plates. Thus:

$$\begin{aligned} E_{0\,x} &= -\frac{\sigma_0}{\epsilon_0}\cos(45^\circ)\hat{x} \\ E_{0\,y} &= \frac{\sigma_0}{\epsilon_0}\sin(45^\circ)\hat{y} \end{aligned}$$

- We need to make a transformation for these two fields. For the component of the electric field strength which is parallel to the velocity vector v, the field for observer S remains the same as for observer S_0. In this case these are the fields in the direction $\hat{x}$. In the direction perpendicular to the velocity we need to add the γ-factor in the transformation. Thus in the present case this holds for the component in the $\hat{y}$ direction. Thus:

$$\begin{aligned} E_x &= E_{0\,x} = -\frac{\sigma_0}{\epsilon_0}\cos(45^\circ)\hat{x} \\ E_{0\,y} &= \gamma E_{0\,y} = \gamma\frac{\sigma_0}{\epsilon_0}\sin(45^\circ)\hat{y} \end{aligned}$$

(4) Question 4
   (a) The Brewster angle is (according to figure) about 54°. Therefore, the angle above the horizon must be $(90-54)^\circ = 36^\circ$.
   (b) The polarization direction will be perpendicular to the plane of incidence. This is therefore parallel to the surface of the lake.
   (c) Polarization is induced by the alignment of the dipoles in a dielectric medium. Aluminum is not a dielectric, therefore the reflected light will be unpolarized (as the incoming light) and with the same intensity.
   (d) He needs to make a setup where the sun-light is scattered over an angle of 90°.
   (e) The incoming light is scattered by the air molecules. These molecules are polarized by the incoming electric field vector and therefore start to oscillate with a frequency equal to the frequency of the incoming light. Therefore, the accelerated charges radiate. The radiation field is maximum at an angle of 90° with respect to the acceleration vector. In a plane perpendicular to the incoming sunlight and hitting the observer, the light will be partially polarized.



For further explanation see minute 40 and further on the web cast of
http://theopenacademy.com/content/lecture-30-polarizers-maluss-law-brewster-angle-and-polarization-relectionscattering
As an alternative, you can consider to use light reflected by water drops in the atmosphere: a.k.a as the rainbow.

A Student's Guide to **Maxwell's Equations**

Home > Chapter 4 Problems

Problem 4.10

To directly measure the displacement current, researchers use a time-varying voltage to charge and discharge a circular parallel-plate capacitor. Find the displacement current density and electric field as a function of time that would produce a magnetic field given by

$$\vec{B} = \frac{r\omega\Delta V\cos(\omega t)}{2d(c^2)}\hat{\varphi}.$$

where *r* is the distance from the center of the capacitor, *ω* is the angular frequency of the applied voltage ΔV, *d* is the plate spacing, and *c* is the speed of light.

Give me a HINT or show me THE FULL SOLUTION! Full Solution Go

5

The differential form of the Ampere-Maxwell law relates the curl of the magnetic field to the conduction current density $\vec{J}$ and the displacement current density ($\varepsilon_0 \partial\vec{E}/\partial t$):

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

In this case, there is no conduction current between the plates of the capacitor, so $\vec{J} = 0$ and

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The displacement current density is thus

$$\varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B})$$

Since $\vec{B}$ has only a ϕ-component, this is

$\vec{J}_D$

$$\varepsilon_0 \frac{\partial \bar{E}}{\partial t} = \frac{1}{\mu_0 r} \frac{\partial}{\partial r}\left[r \frac{r\,\omega\,\Delta V \cos(\omega t)}{2d(c^2)} \right] \hat{z}$$

$$= \frac{1}{\mu_0 r}\left[\frac{\omega\,\Delta V \cos(\omega t)}{2d(c^2)} \right] \frac{\partial (r^2)}{\partial r} \hat{z}$$

5a

$$= \frac{\omega\,\Delta V \cos(\omega t)}{\mu_0\, d(c^2)} \hat{z} \quad \text{(answer)}$$

To find the electric field, integrate over time:

$$\bar{E}(t) = \int_0^t \frac{\omega\,\Delta V \cos(\omega t)}{\mu_0\, \varepsilon_0\, d(c^2)} \hat{z}\, dt$$

Thus

$$\bar{E}(t) = \frac{1}{\omega}\left[\frac{\omega\,\Delta V \sin(\omega t)}{\mu_0\, \varepsilon_0\, d(c^2)} \right] \hat{z}$$

5b

$$= \frac{\Delta V \sin(\omega t)}{\mu_0\, \varepsilon_0\, d(c^2)} \hat{z} \quad \text{(answer)}$$

$$= \frac{\Delta V \sin(\omega t)}{d} \hat{z}$$

question 6

Problem 7.35

6 (a) $\mathbf{E} = \frac{\sigma(t)}{\epsilon_0}\hat{\mathbf{z}}$; $\sigma(t) = \frac{Q(t)}{\pi a^2} = \frac{It}{\pi a^2}$; $\boxed{\frac{It}{\pi\epsilon_0 a^2}\hat{\mathbf{z}}.}$ 6c

6b (b) $I_{d_{\text{enc}}} = J_d \pi s^2 = \epsilon_0 \frac{dE}{dt}\pi s^2 = \boxed{I\frac{s^2}{a^2}.}$ $\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{d_{\text{enc}}} \Rightarrow B\,2\pi s = \mu_0 I \frac{s^2}{a^2} \Rightarrow \mathbf{B} = \boxed{\frac{\mu_0 I}{2\pi a^2} s\,\hat{\boldsymbol{\phi}}.}$

Problem 8.2

6a (a) $\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{z}}$; $\sigma = \frac{Q}{\pi a^2}$; $Q(t) = It \Rightarrow \mathbf{E}(t) = \boxed{\frac{It}{\pi\epsilon_0 a^2}\hat{\mathbf{z}}.}$

$$B\,2\pi s = \mu_0\epsilon_0 \frac{\partial E}{\partial t}\pi s^2 = \mu_0\epsilon_0 \frac{I\pi s^2}{\pi\epsilon_0 a^2} \Rightarrow \mathbf{B}(s,t) = \boxed{\frac{\mu_0 I s}{2\pi a^2}\hat{\boldsymbol{\phi}}.}$$ 6c

6d (b) $u_{\text{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \frac{1}{2}\left[\epsilon_0\left(\frac{It}{\pi\epsilon_0 a^2}\right)^2 + \frac{1}{\mu_0}\left(\frac{\mu_0 I s}{2\pi a^2}\right)^2\right] = \boxed{\frac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2 + (s/2)^2\right].}$

6e $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0}\left(\frac{It}{\pi\epsilon_0 a^2}\right)\left(\frac{\mu_0 I s}{2\pi a^2}\right)(-\hat{\mathbf{s}}) = \boxed{-\frac{I^2 t}{2\pi^2\epsilon_0 a^4}s\,\hat{\mathbf{s}}.}$

6e $\frac{\partial u_{\text{em}}}{\partial t} = \frac{\mu_0 I^2}{2\pi^2 a^4}2c^2 t = \frac{I^2 t}{\pi^2\epsilon_0 a^4}$; $-\boldsymbol{\nabla}\cdot\mathbf{S} = \frac{I^2 t}{2\pi^2\epsilon_0 a^4}\boldsymbol{\nabla}\cdot(s\,\hat{\mathbf{s}}) = \frac{I^2 t}{\pi^2\epsilon_0 a^4} = \frac{\partial u_{\text{em}}}{\partial t}.$ ✓

6f (c) $U_{\text{em}} = \int u_{\text{em}} w 2\pi s\, ds = 2\pi w \frac{\mu_0 I^2}{2\pi^2 a^4}\int_0^b \left[(ct)^2 + (s/2)^2\right] s\, ds = \frac{\mu_0 w I^2}{\pi a^4}\left[(ct)^2\frac{s^2}{2} + \frac{1}{4}\frac{s^4}{4}\right]\Bigg|_0^b$

6g

$= \boxed{\frac{\mu_0 w I^2 b^2}{2\pi a^4}\left[(ct)^2 + \frac{b^2}{8}\right].}$ Over a surface at radius b: $P_{\text{in}} = -\int \mathbf{S}\cdot d\mathbf{a} = \frac{I^2 t}{2\pi^2\epsilon_0 a^4}\left[b\,\hat{\mathbf{s}}\cdot(2\pi b w\,\hat{\mathbf{s}})\right] = \boxed{\frac{I^2 w t b^2}{\pi\epsilon_0 a^4}.}$

6h $\frac{dU_{\text{em}}}{dt} = \frac{\mu_0 w I^2 b^2}{2\pi a^4}2c^2 t = \frac{I^2 w t b^2}{\pi\epsilon_0 a^4} = P_{\text{in}}.$ ✓ (Set $b = a$ for *total*.)

Problem 8.3

The force is clearly in the z direction, so we need

1